Vertex Algebras

Notation:
- V is a supervector space ie) $V = V_0 \oplus V_1$
- A field $a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$, where each
  in super vector space
  $a_{(n)} \in \mathrm{End}(V)$ which should satisfy
  a superalgebra $a_{(n)} v = 0 \quad \forall v \in V$, any $n \gg 0$
- a is said to have parity $p(a)$ if $a_{(n)} V_\alpha \subset V_{\alpha + p(a)}$
  $\forall \alpha \in \mathbb{Z}/2\mathbb{Z} \quad \forall n \in \mathbb{Z}$

A Vertex Algebra consists of the following data:
- The space of states: V
- The vacuum vector: $|0\rangle \in V_0$
  (Note: vacuum vector cannot be degenerate)
- State-Field correspondence: A parity-preserving linear map from V to the space of fields

$$\begin{array}{ccc} a & \longmapsto & Y(a.z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^n \\ \cap & & \\ V & & a(z) \end{array}$$

Satisfying the following:

supercommutator (T has parity 0 so doesn't matter)

1) (Translation covariance): $[T, a_{(n)}] = -n\, a_{(n-1)}$

where $T \in \mathrm{End}(V)$. $\quad T(a) = a_{(-2)}|0\rangle$

2) (Vacuum)

$$|0\rangle_{(n)} = \delta_{n,-1} \quad (\text{implies } T|0\rangle = 0)$$

$$a_{(n)}|0\rangle = 0 \quad \text{for } n \geq 0$$

$$a_{(-1)}|0\rangle = a$$

3) (Locality)

$$(z-w)^N \underbrace{[a(z), b(w)]}_{a(z)b(w) - (-1)^{p(a)p(b)} b(w)a(z)} = 0 \quad \text{for } N \gg 0$$

Exercise:

$$\frac{T^n a}{n!} = a_{-n-1}|0\rangle$$

$$\Updownarrow$$

$$a(z)|0\rangle = e^{zT}(a)$$

Holomorphic Vertex Algebra

A Vertex Algebra is called holomorphic if $a_{(n)} = 0 \quad \forall n \geq 0 \quad \forall a \in V$.

In this case, $a(z) = \sum_{n \in \mathbb{N}} a_{-n-1} z^n$

Claim: A holomorphic vertex algebra $\cong$ a commutative superalgebra + Derivation

Proof:

$\Rightarrow$ - Direction

Define a product on V

$$ab = a_{(-1)} b$$

$|0\rangle$ is the unit

$$V_\alpha V_\beta \subset V_{\alpha+\beta}$$

Now, Check assoc. + comm.:

Observation: The algebra of formal power series has no zero divisor $\Rightarrow$

(Locality condition) $\Longleftrightarrow [a(z), b(w)] = 0$

ie) $a(z) b(w) = (-1)^{p(a)p(b)} b(w) a(z)$

Comparing LHS & RHS term by term:

$$\Rightarrow a_{(-n)} b_{(-m)} = (-1)^{p(a)p(b)} b_{(-m)} a_{(-n)}$$

$\times$ by c

$$\Rightarrow a(bc) = (-1)^{p(a)p(b)} b(ac)$$

$c = |0\rangle$

$$ab = (-1)^{p(a)p(b)} ba$$

gives supercommutativity.

$$\Rightarrow a(cb) = (ac)b \quad \text{aka Associativity}$$

Use supercomm. & commute elements

and T gives a derivation on V:

Translation covariance $\Rightarrow T a_{(n)} - a_{(n)} T = -n\, a_{(n-1)}$

Let $n = -1$

$$T a_{(-1)} - a_{(-1)} T = a_{(-2)}$$

$$T(ab) - a(Tb) = \underbrace{a_{(-2)} b}$$

Need to show this is $= (Ta)b$

$$(Ta)b \overset{\text{by defn. of } T}{=} (a_{(-2)}|0\rangle) \cdot b \overset{\text{supercomm.}}{=} (-1)^{p(a)p(b)} b_{(-1)} (a_{(-2)}|0\rangle)$$

$$\overset{\text{Locality}}{=} \; a_{(-2)} b_{(-1)} |0\rangle$$

$$= a_{(-2)} b \quad //$$

$\Leftarrow$ – Direction:

Given an assoc., supercomm. Superalgebra V and a derivation T, we need to construct a holomorphic vertex algebra:

Set:

- Space of states $= V$
- $|0\rangle = 1_V$
- Let the T be the derivation
- $a(z) b = (e^{zT} a) b$

$\Updownarrow$ Equivalently

$$a_{(-n-1)} b = \frac{T^n a}{n!} b$$

Now check whether these satisfy the axioms of a holomorphic vertex algebra: Exercise. //

$\square$

Chapter-2

Calculus of Formal distributions

A formal distribution in the indeterminates $z_1, z_2, \ldots$ with the values in a vector space U is a formal expression

$$\sum_{\substack{m_1, m_2, \ldots \\ \in \mathbb{Z}}} a_{m_1, m_2, \ldots} z_1^{m_1} z_2^{m_2} \cdots\cdots$$

where $a_{m_1, m_2, \ldots} \in U$

Given $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n$, its residue $\operatorname{Res}_z a(z) := a_{-1}$

Remark:

$$\operatorname{Res}_z \left(\partial a(z)\, b(z) \right) = - \operatorname{Res}_z \left(a(z)\, \partial b(z) \right)$$

A Formal delta function: (A formal distribution in z & w with values in $\mathbb{C}$):

$$\delta(z-w) := z^{-1} \sum_{n \in \mathbb{Z}} z^n w^{-n}$$

Prop.

i) $\forall\, f(z) \in U[[z, z^{-1}]]$, we have

$$\mathrm{Res}_z\, f(z)\,\delta(z-w) = f(w)$$

2) $\delta(z-w) = \delta(w-z)$

3) $\partial_z^j \delta(z-w) = (-\partial_w)^j \delta(z-w)$

4) $(z-w)\,\partial_w^{(j+1)} \delta(z-w) = \partial_w^{(j)} \delta(z-w)$ $\quad A^{(j)} = \frac{A^j}{j!}$

5) $(z-w)^{j+1}\, \partial_w^{(j)}\, \delta(z-w) = 0$

<u>Proof:</u>

By direct calculation from the defn. of $\delta(z-w)$ above $\square$

Consider the map

$$\Pi : U[[z,z^{-1},w,w^{-1}]] \longrightarrow U[[z,z^{-1},w,w^{-1}]]$$

$$a(z,w) \longmapsto \sum_{j\in\mathbb{N}} \mathrm{Res}_z \left(a(z,w)(z-w)^N\right) \partial_w^{(j)} \delta(z-w)$$

This map has the following properties:

<u>Prop:</u>

1) $\Pi^2 = \Pi$

2) $\mathrm{Ker}\,\Pi = \{a(z,w) \mid a(z,w) = a(z,w)^{+z}\}$ where,

$$a(z,w)^{+z} = \sum_{\substack{m\in\mathbb{Z}_+ \\ n\in\mathbb{Z}}} a_{mn} z^m w^n$$

1) + 2) $\Rightarrow$ $\forall\, a(z,w) = \prod a(z,w) + b(z,w)$

has this decomposition

Rough sketch of
Proof: (Kinda direct computation)

1) If $a(z,w) = \sum_{j\in\mathbb{N}} c^j(w)\, \partial_w^{(j)} \delta(z-w)$, then

$$c^j(w) = \mathrm{Res}_z \left(a(z,w)(z-w)^j \right)$$

$$a(z,w)(z-w)^n = \sum_{j\in\mathbb{N}} c^j(w)\, \mathrm{Res}_z \underbrace{\left(\partial_w^{(j)} \delta(z-w)(z-w)^n\right)}_{=0 \text{ if } n>j}$$

If $n=j$, then we get

$$c^n(w)$$

If $n<j$, then

$\partial_w^{(j-n)} \delta(z-w)$, it's still zero

2) Suppose $a(z,w) = \sum_{n\in\mathbb{Z}} a_n(w) z^n$

Suppose $\sum_{n\in\mathbb{Z}} a_n(w) \operatorname{Res}_z z^n (z-w)^n = 0 \quad \forall n \in \mathbb{N}$

$\Rightarrow$ Each term is zero $\because$ we can't cancel terms.
So pick a term & equate it to zero

$\square$

Corollary:

If $a(z,w)(z-w)^N = 0$, then
$$a(z,w) = \sum_{j=0}^{N-1} \operatorname{Res}_z \left(a(z,w)(z-w)^j \right) \partial_w^{(j)} \delta(z-w)$$

Proof:

From prev. prop.,
$$a(z,w) = \sum_{j\in\mathbb{N}} \operatorname{Res}_z \left(a(z,w)(z-w)^j \right) \partial_w^{(j)} \delta(z-w) + b(z,w)$$

$\times (z-w)^N$ on both sides,

$$\underbrace{a(z,w)(z-w)^N}_{\substack{=0 \text{ by} \\ \text{assumption,}}} = \sum_{j\in\mathbb{N}} \mathrm{Res}_z \left(a(z,w)(z-w)^j\right) \underbrace{\partial_w^{(j)}\delta(z-w)(z-w)^N}_{\substack{\text{vanishes} \\ \text{for } j>N \text{ (or} \\ j<N)}}$$

$$+\, b(z,w)(z-w)^N$$

For $j=N$, $\quad \mathrm{Res}_z \left(a(z,w)\underbrace{(z-w)^N}\right)$

$$\underbrace{\qquad\qquad\qquad}_{=0}$$

$$\Rightarrow b(z,w)(z-w)^N = 0$$

$\square$

Locality

Defn: $a(z), b(w)$ formal distri with values in U.
These are called *local* if

$$(z-w)^N [a(z), b(w)] = 0 \quad \text{for } N \gg 0$$

Notations:

1) $$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \quad \text{where } a_{(n)} = a_{-n-1}$$

$$= \mathrm{Res}_z\, a(z) z^n$$

$$\| $$

$$\sum_{n\in\mathbb{Z}} a_n z^n$$

2) $a(z)_- = \sum_{n \geq 0} a_{(n)} z^{-n-1}$

$a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1}$

3) $:a(z)b(w): \; = a(z)_+ b(w) + (-1)^{p(a)p(b)} b(w) a(z)_-$

$= a(z)\,b(w)$, if $a(z)$ is holomorphic.

We have: $a(z)b(w) = [a(z)_-, b(w)] + :a(z)b(w):$

$(-1)^{p(a)p(b)} b(w)a(z) = -[a(z)_+, b(w)] + :a(z)b(w):$

For each $n \in \mathbb{N}$, define

$$a(z)_n b(w) = \mathrm{Res}_z \, [a(z), b(w)](z-w)^n$$

Thm:

Let $c^j(w) = a(z)_j b(w)$, then the following properties $\simeq$ locality condition.

(i) $[a(z), b(w)] = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(w-j)\, c^j(w)$

(ii) $$[a(z)_-, b(w)] = \sum_{j=0}^{N-1} i_{z,w} \frac{1}{(z-w)^{j+1}} c^j(w)$$

$$[a(z)_+, b(w)] = \sum_{j=0}^{N-1} i_{w,z} \frac{1}{(z-w)^{j+1}} c^j(w)$$

(iii) $$a(z)\, b(w) = \sum_{j=0}^{N-1} i_{z,w} \frac{1}{(z-w)^{j+1}} c^j(w) + :a(z)b(w):$$

✓ Defn. of OPE

$$(-1)^{p(a)p(b)}\, b(w)\,a(z) = \sum_{j=0}^{N-1} i_{w,z} \frac{1}{(z-w)^{j+1}} c^j(w) + :a(z)b(w):$$

$$i_{z,w} \frac{1}{(z-w)^{j+1}} = \sum_{m=0}^{\infty} \binom{m}{j} z^{-m-1} w^{m-j}$$

$$i_{w,z} \frac{1}{(z-w)^{j+1}} = -\sum_{m=-1}^{\infty} \binom{m}{j} z^{-m-1} w^{m-j}$$

By abuse of notation,

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{a(z)_j\, b(w)}{(z-w)^{j+1}}$$

Prop.

1) $\partial a(z)_{(n)} b(w) = -n\, a(z)_{(n-1)} b(w)$

$$\partial_w \left(a(z)_n b(w)\right) = a(z)_n \partial b(w) + \partial a(z)_{(n)} b(w)$$

2) For any $a(z), b(w)$ mutually local,

$$a(z)_{(n)} b(w) = (-1)^{p(a)p(b)} \sum_{j=0}^{N-1} (-1)^{j+n+1} \partial^{(j)} \left(b(w)_{(n+j)} a(z)\right)$$

3) For three formal distri.

$$a(z)_{(m)} \left(b(w)_{(n)} c(u)\right) = \sum_{j=0}^{m} \binom{m}{j} \left(a(z)_{(j)} b(w)\right)_{(m+n-j)} c(u)$$
$$+ (-1)^{p(a)p(b)} b(w)_{(n)} \left(a(z)_{(m)} c(u)\right)$$

Corollary:

$[a_{(0)}, b(w)] = 0$ iff $a(z)_{(0)} b(w) = 0$;

if $a(z)$ is odd, then $a_{(0)}^2 = 0$ iff

$$\mathrm{Res}_w\, a(z)_{(0)} a(w) = 0$$